OCR

RECOGNISING ACHIEVEMENT

General Certificate of Education
Advanced Supplementary (AS) and Advanced Level

former Oxford and Cambridge modular syllabus

MEI STRUCTURED MATHEMATICS **5510/1**

Differential Equations (Mechanics 4)

Thursday **21 JUNE 2001** Afternoon 1 hour 20 minutes

Additional materials:
Answer paper
Graph paper
Students' Handbook

TIME 1 hour 20 minutes

INSTRUCTIONS TO CANDIDATES

Write your name, Centre number and candidate number in the spaces provided on the answer paper/answer booklet.

Answer any **three** questions.

Write your answers on the separate answer paper provided.

If you use more than one sheet of paper, fasten the sheets together.

INFORMATION FOR CANDIDATES

The approximate allocation of marks is given in brackets [ ] at the end of each question or part question.

You are advised that an answer may receive no marks unless sufficient detail of the working is shown on the answer paper to indicate that a correct method is being used.

Take $g = 9.8\ \text{m s}^{-2}$ unless otherwise instructed.

This question paper consists of 3 printed pages and 1 blank page.

1 The current in an electrical circuit consisting of an inductor, resistor and capacitor in series with an exponentially decaying power source is modelled by the equation

$$10\frac{d^2I}{dt^2} + 2k\frac{dI}{dt} + I = 18e^{-\frac{1}{10}t},$$

where I is the current in amperes and t is the time in seconds after the power source is switched on. The value of the positive constant k can be adjusted by varying the resistance.

In the case $k = 1$,

(i) find the general solution, [8]

(ii) find the particular solution for which $\frac{dI}{dt} = I = 0$ when $t = 0$. [4]

The resistance is adjusted so that the current will not oscillate.

(iii) Show that the minimum value of k required to achieve this is $\sqrt{10}$. [2]

With $k = \sqrt{10}$ and the power source removed, the equation modelling the current is

$$10\frac{d^2I}{dt^2} + 2\sqrt{10}\frac{dI}{dt} + I = 0,$$

with $I = 4$ and $\frac{dI}{dt} = a$ when $t = 0$.

(iv) Find the general solution of this equation. Determine the range of values of a for which the current will not be zero at any finite time. [6]

2 The solution is sought for the equations

$$\frac{dx}{dt} = 4x - y + 50\sin t$$

$$\frac{dy}{dt} = 6x - 3y + 50\cos t.$$

(i) Eliminate y from the equations to show that

$$\frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 150\sin t.$$ [5]

(ii) Find the general solution for x in terms of t. [8]

It is given that when $t = 0$, $x = 0$ and $y = 15$.

(iii) Find the particular solutions for x and y in terms of t. [7]

3 The solution is sought for the differential equation

$$\left(1+x^2\right)\frac{dy}{dx}-\frac{4x^3y}{1-x^2}=1 \qquad (-1<x<1).$$

(i) Solve the equation to show that

$$y=\frac{k+3x-x^3}{3\left(1-x^4\right)}, \qquad (*)$$

where k is an arbitrary constant. [8]

(ii) Find the particular solution in each of the cases

(A) $y = 1$ when $x = 0$,

(B) $y = 0$ when $x = 0$.

In each case, draw a sketch graph of the solution for $-1 < x < 1$. [8]

(iii) Find the value of k in (*) for which y tends to a finite limit as x tends to 1. [4]

4 A rocket of constant mass m is launched vertically from rest on the earth's surface. The forces on the rocket are a thrust from its engines and the force of gravity. The thrust has magnitude Am, where A is a constant. The magnitude of the force due to gravity on the rocket when it is a distance x above the surface of the earth is $\frac{R^2mg}{(R+x)^2}$, where R is the radius of the earth and g is the acceleration due to gravity at the earth's surface.

The rocket has velocity v at a distance x above the earth's surface.

(i) Write down a differential equation connecting v and x. Solve the equation to show that

$$v^2=2Ax-\frac{2Rgx}{R+x}. \qquad [9]$$

The thrust on the rocket ceases when $x = a$. Subsequently, the only force acting is due to gravity.

(ii) Solve an appropriate differential equation to show that, for the motion after the thrust ceases,

$$v^2=\frac{2R^2g}{R+x}+2(Aa-Rg). \qquad [6]$$

(iii) Determine the range of values of a for which the rocket will fall back to earth. [3]

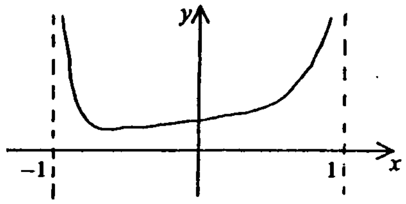
(iv) If the rocket does not fall back to earth, find its terminal velocity. [2]

Mark Scheme

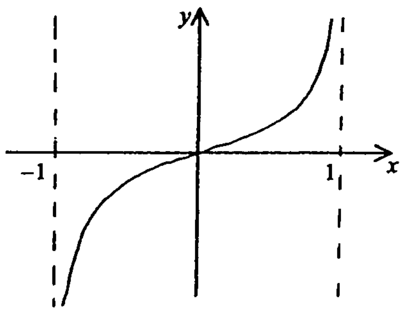
1.(i)	$10\lambda^2+2\lambda+1=0$	M1	
	$\lambda=-\frac{1}{10}\pm\frac{3}{10}i$	A1	
	CF $I=e^{-\frac{1}{10}t}(A\cos\frac{3}{10}t+B\sin\frac{3}{10}t)$	F1	
	PI $I=ae^{-\frac{1}{10}t}$	B1	correct form
	$I'=-\frac{a}{10}e^{-\frac{1}{10}t}$, $I''=\frac{a}{100}e^{-\frac{1}{10}t}$	M1	differentiate and substitute
	$(\frac{a}{10}-\frac{2a}{10}+a)e^{-\frac{1}{10}t}=18e^{-\frac{1}{10}t}$	M1	compare coefficients
	$\Rightarrow a=20$	A1	
	$I=e^{-\frac{1}{10}t}(20+A\cos\frac{3}{10}t+B\sin\frac{3}{10}t)$	F1	CF + PI
(ii)	$0=20+A\Rightarrow A=-20$	B1	condition on I
	$I'=-\frac{1}{10}e^{-\frac{1}{10}t}(20+A\cos\frac{3}{10}t+B\sin\frac{3}{10}t)$ $+e^{-\frac{1}{10}t}(-\frac{3}{10}A\sin\frac{3}{10}t+\frac{3}{10}B\cos\frac{3}{10}t)$	M1	differentiate (product rule)
	$0=-\frac{1}{10}(20-20)+\frac{3}{10}B\Rightarrow B=0$	M1	condition on I'
	$I=20e^{-\frac{1}{10}t}(1-\cos\frac{3}{10}t)$	A1	
(iii)	$10\lambda^2+2k\lambda+1$ has repeated root $\Rightarrow(2k)^2=4\times10$	M1	recognising repeated root
	$\Rightarrow k=\sqrt{10}$	E1	
(iv)	repeated root $\lambda=-\frac{1}{\sqrt{10}}$	M1	repeated root
	CF $I=(C+Dt)e^{-\frac{1}{\sqrt{10}}t}$	A1	
	$t=0, I=4\Rightarrow C=4$	M1	condition on I
	will never reach zero if $D>0$	M1	
	$I'=De^{-\frac{1}{\sqrt{10}}t}+(C+Dt)(-\frac{1}{\sqrt{10}})e^{-\frac{1}{\sqrt{10}}t}$		
	$a=D-\frac{C}{\sqrt{10}}$	M1	differentiate and substitute
	$D>0\Rightarrow a>-\frac{4}{\sqrt{10}}$	A1	cao

2.(i)	$\frac{d^2x}{dt^2} = 4\frac{dx}{dt} - \frac{dy}{dt} + 50\cos t$	M1	
		A1	
	$= 4\frac{dx}{dt} - 6x + 3y - 50\cos t + 50\cos t$	M1	substitute for $\frac{dy}{dt}$
	$= 4\frac{dx}{dt} - 6x + 3(4x + 50\sin t - \frac{dx}{dt})$	M1	substitute for y
	$\Rightarrow \frac{d^2x}{dt^2} - \frac{dx}{dt} - 6x = 150\sin t$	E1	
(ii)	$\lambda^2 - \lambda - 6 = 0$	M1	attempt to solve auxiliary equation
	$\lambda = 3$ or -2	A1	
	CF $x = Ae^{3t} + Be^{-2t}$	F1	
	PI $x = a\sin t + b\cos t$	B1	correct form
	$\frac{dx}{dt} = a\cos t - b\sin t$, $\frac{d^2x}{dt^2} = -a\sin t - b\cos t$		
	$(-a\sin t - b\cos t) - (a\cos t - b\sin t)$ $-6(a\sin t + b\cos t) = 150\sin t$	M1	differentiate twice and substitute
	$-7a + b = 150$ and $-a - 7b = 0$	M1	compare coefficients
	$a = -21,\ b = 3$	A1	
	$x = -21\sin t + 3\cos t + Ae^{3t} + Be^{-2t}$	F1	PI + CF
(iii)	$0 = 3 + A + B$	B1	
	$y = 4(-21\sin t + 3\cos t + Ae^{3t} + Be^{-2t})$	M1	$\frac{dx}{dt}$
	$+50\sin t - (-21\cos t - 3\sin t + 3Ae^{3t} - 2Be^{-2t})$	M1	attempt at y in terms of t
	$15 = 33 + A + 6B$	A1	
	[or $-21 + 3A - 2B = -15$]		
	$A = 0,\ B = -3$	M1	
	$x = -21\sin t + 3\cos t - 3e^{-2t}$	A1	
	$y = -31\sin t + 33\cos t - 18e^{-2t}$	A1	

3.(i)	$\frac{dy}{dx} - \frac{4x^3}{1-x^4}y = \frac{1}{1+x^2}$	M1	rearranging
	$I = \exp(\int \frac{-4x^3}{1-x^4}dx)$	M1	attempt to find integrating factor
	$= 1 - x^4$	A1	
	$(1-x^4)\frac{dy}{dx} - 4x^3y = 1 - x^2$	M1	multiply by IF
	$\frac{d}{dx}((1-x^4)y) = 1 - x^2$	M1	
	$(1-x^4)y = x - \frac{x^3}{3} + A$	M1	integrating
		A1	
	$y = \frac{k+3x-x^3}{3(1-x^4)}$	E1	
(ii)(A)	$1 = \frac{k}{3}$	M1	condition
	$y = \frac{3+3x-x^3}{3(1-x^4)}$	A1	
		B1	shape (drawn for $-1 < x < -1$ only)
		B1	asymptotes clearly shown



(B)	$0 = \frac{k}{3}$	M1	condition
	$y = \frac{3x-x^3}{3(1-x^4)}$	A1	
		B1	shape (drawn for $-1 < x < -1$ only)
		B1	asymptotes clearly shown



(iii)	denominator $= 3(1-x)(1+x)(1+x^2)$	B1	$(1-x)$ factor or behaviour as $x \to 1$
	$\therefore$ numerator needs factor $1-x$	M1	or equivalent
	$\Rightarrow k + 3 - 1^3 = 0$	M1	
	$\Rightarrow k = -2$	A1	

4.(i)	$mv\frac{dv}{dx} = Am - \frac{R^2mg}{(R+x)^2}$	M1	N2L
		A1	LHS
		A1	RHS
	$\int v\,dv = \int(A - \frac{R^2g}{(R+x)^2})dx$	M1	separate variables and integrate
	$\frac{v^2}{2} = Ax + \frac{R^2g}{R+x} + C_1$	A1	LHS
		A1	RHS
	$x = 0, v = 0 \Rightarrow C_1 = -\frac{R^2g}{R} = -Rg$	M1	using initial condition
		A1	
	$v^2 = 2Ax + \frac{2R^2g}{R+x} - 2Rg$		
	$v^2 = 2Ax - \frac{2Rgx}{R+x}$	E1	
(ii)	$x = a \Rightarrow v^2 = 2Aa - \frac{2Rga}{R+a}$	M1	finding initial condition
		A1	
	$v\frac{dv}{dx} = -\frac{R^2g}{(R+x)^2}$	M1	DE with no thrust
	$\frac{v^2}{2} = \frac{R^2g}{R+x} + C_2$	A1	
	$x = a,\ Aa - \frac{Rga}{R+a} = \frac{R^2g}{R+a} + C_2$	M1	using initial condition
	$\Rightarrow C_2 = Aa - Rg$		
	$\Rightarrow v^2 = \frac{2R^2g}{R+x} + 2(Aa - Rg)$	E1	
(iii)	$v^2 = 0$ for some $x \geq a$	M1	
	$\Rightarrow Aa - Rg < 0$	M1	
	$\Rightarrow a < \frac{Rg}{A}$	A1	
(iv)	as $x \to \infty,\ \frac{2R^2g}{R+x} \to 0$	M1	
	$v \to \sqrt{2(Aa - Rg)}$	A1	

Examiner’s Report

Differential Equations (Mechanics 4) (5510)

General Comments

There were many very good responses to this paper. Most candidates did questions 1 and 2 and did them well. Candidates then mainly attempted question 3. Question 4 was not a popular choice, even though the differential equations generated were the easiest on the paper. This may be because candidates had to formulate the differential equations instead of being given them.

Comments on Individual Questions

Question 1 (Current in an electric circuit: Second order equation)

The vast majority of candidates knew how to solve the equation but some made arithmetical and algebraic slips. Many candidates were able to calculate the minimum value of k for the solution not to oscillate, but some merely showed that the given value gave a non-oscillatory solution, without showing that it was minimal. Candidates sometimes had difficulty dealing with the repeated root of the auxiliary equation in the last part, but many were able to find the general solution correctly. Only a minority of candidates were able correctly to determine the range of values of a for which the current would not become zero. Most attempts left t in the answer.

(i) $I = e^{-\frac{1}{10}t}(20 + A\cos\frac{3}{10}t + B\sin\frac{3}{10}t)$; (ii) $I = 20e^{-\frac{1}{10}t}(1 - \cos\frac{3}{10}t)$; (iv) $I = (C + Dt)e^{-\frac{1}{\sqrt{10}}t}$, $a > -4/\sqrt{10}$

Question 2 (Simultaneous equations)

This question was often very well done, marred only by simple errors. However, some candidates would do well to practise their strategy of elimination of one variable. This was often done very succinctly, but also often done in a very long-winded manner, almost stumbling on the final solution by accident. Some candidates changed signs without justification to produce the given result, whereas when a sign error is spotted it should be traced back and corrected. Another common problem was that some candidates having got a solution for x, created a differential equation for y and either were unable to reconcile the arbitrary constants, or did not realise that they cannot simply use the same arbitrary constants as for x. With this type of question, candidates should always look to find y from the differential equation not containing $\mathrm{d}y/\mathrm{d}t$, as, if this is possible, it is a much simpler process.

$$\text{(ii)}\ x = 3\cos t - 21\sin t + Ae^{3t} + Be^{-2t};\ \text{(iii)}\ x = 3\cos t - 21\sin t - 3e^{-2t}, y = 33\cos t - 31\sin t - 18e^{-2t}$$

Question 3 (Integrating Factor Method)

Although some candidates tried to separate variables and some omitted the first part of this question, most realised that the integrating factor method was necessary to solve the equation. However there were many errors made, particularly with signs. It was pleasing to see some candidates trace back sign errors and correct them completely, but most sign errors were conveniently ignored or signs were changed without justification.

The particular solutions presented few problems, except some candidates forgot to state the solution as asked and merely stated the value of k. Although there were some very good graphs drawn, there were also some very sloppy sketches. A detailed analysis of the solutions was not expected (although quite welcome if candidates have sufficient time) and candidates are at liberty to use graphical calculators if they have them. However a simple copy of a graphical calculator screen without intelligent interpretation in these cases gained no credit. Candidates were expected to clearly show the asymptotic behaviour at 1 and –1. Candidates were also expected to sketch only between –1 and 1 as instructed in the question, as also indicated by the original domain of the equation and as also required for mathematical validity. The last part was sometimes well done, but many candidates were unclear what method they were using and almost stumbled across the correct solution without justifying it.

$$\text{(ii)}\ y = \frac{3 + 3x - x^3}{3(1 - x^4)},\quad y = \frac{3x - x^3}{3(1 - x^4)};\qquad \text{(iii)}\ -2.$$

Question 4 (Motion of a rocket: Separable variables equations)

The minority of candidates who attempted this question were normally able to find the correct expression for acceleration, but some were not able to use this to write a correct equation connecting velocity and displacement but not time. There were a fair number of correct solutions, but some candidates jumped straight to the given answer when integrating, rather than using the standard method when integrating and dealing with the constant properly. Candidates must take particular care to show necessary working when an answer is given.

Candidates often had problems deciding on the conditions needed to determine the constant of integration in the second phase of the motion, but there were a number of good solutions.

Most candidates set v^2 either equal to or less than zero when considering whether the rocket would fall back to earth, but many were unable to find a range of values of a independent of x. Most candidates could find the terminal velocity with little problem, but this did not seem to give fresh insight into the previous answer.

$$\text{(iii)}\ a < Rg/A;\qquad \text{(iv)}\ \sqrt{2(Aa - Rg)}.$$